Sanwa Bank California

ASSET ALLOCATION

A Primer

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ASSET ALLOCATION
CASE 1: BASIC OPTIMIZATION

$$\underset{w_S, w_B, w_C}{\text{MAX}} \qquad \hat{R}_P = w_S \hat{R}_S + w_B \hat{R}_B + w_C \hat{R}_C$$

Subject to:

$$\hat{\sigma}_P^2 \leq \overline{\sigma}_P^2$$

$$w_S + w_B + w_c \leq LEVR$$

$$w_S, w_B, w_C \geq 0$$

WHERE:

$\hat{R}_P$ = The Expected Gross Return on the Portfolio

$\hat{R}_S, \hat{R}_B, \hat{R}_C$ = The Expected Returns for the Stock, Bond, and Cash Asset Classes

w_S, w_B, w_C = The portfolio weights for the Stock, Bond, and Cash Asset Classes

$$\hat{\sigma}_P^2 = w_S^2 Var(\hat{R}_S) + w_B^2 Var(\hat{R}_B) + w_C^2 Var(\hat{R}_C) + 2 w_S w_B Cov(\hat{R}_S, \hat{R}_B) + 2 w_S w_C Cov(\hat{R}_S, \hat{R}_C) + 2 w_B w_C Cov(\hat{R}_B, \hat{R}_C)$$

= The Expected Variance of the Portfolio

$\overline{\sigma}_P^2$ = The Maximum Portfolio Variance to be Tolerated

LEVR = The leverage factor of the portfolio (100% = unleveraged, more = leverage)

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CASE 2: BASIC OPTIMIZATION
with Hard Constraints

$$\text{MAX} \quad \hat{R}_P = w_S \hat{R}_S + w_B \hat{R}_B + w_C \hat{R}_C$$

$$w_S, w_B, w_C$$

Subject to:

$$\hat{\sigma}_P^2 \le \overline{\sigma}_P^2$$

$$w_S \le STK^+, w_S \ge STK^-$$

$$w_B \le BND^+, w_B \ge BND^-$$

$$w_C \le CSH^+, w_C \ge CSH^-$$

$$w_S \beta_S + w_B \beta_B + w_c \beta_C \le \beta^*$$

$$w_S + w_B + w_c \le LEVR$$

$$w_S, w_B, w_C \ge 0$$

WHERE:

$\hat{R}_P$ = The Expected Gross Return on the Portfolio

$\hat{R}_S, \hat{R}_B, \hat{R}_C$ = The Expected Returns for the Stock, Bond, and Cash Asset Classes

w_S, w_B, w_C = The portfolio weights for the Stock, Bond, and Cash Asset Classes

$$\hat{\sigma}_P^2 = w_S^2 Var(\hat{R}_S) + w_B^2 Var(\hat{R}_B) + w_C^2 Var(\hat{R}_C) + 2 w_S w_B Cov(\hat{R}_S, \hat{R}_B) + 2 w_S w_C Cov(\hat{R}_S, \hat{R}_C) + 2 w_B w_C Cov(\hat{R}_B, \hat{R}_C)$$

= The Expected Variance of the Portfolio

$\overline{\sigma}_P^2$ = The Maximum Portfolio Variance to be Tolerated

β_{ii} = A Factor associated with holding one unit of Stocks, Bonds, or Cash

STK^+, STK^- = Upper and lower Bounds for Stock Holdings

BND^+, BND^- = Upper and lower Bounds for Bond Holdings

CSH^+, CSH^- = Upper and lower Bounds for Cash Holdings

LEVR = The leverage Factor of the Portfolio (100% = no leverage, over 100% = leverage

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CASE 3: TURNOVER-SENSITIVE OPTIMIZATION

$$\text{MAX} \quad \hat{r}_P = w_S \hat{R}_S + w_B \hat{R}_B + w_C \hat{R}_C - \left|w_S - w_S^*\right| \cdot \tau_S - \left|w_B - w_B^*\right| \cdot \tau_B - \left|w_C - w_C^*\right| \cdot \tau_C$$

$$w_S, w_B, w_C$$

Subject to:

$$\hat{\sigma}_P^2 \le \overline{\sigma}_P^2$$

$$w_S + w_B + w_C \le LEVR$$

$$\left|w_S - w_S^*\right| + \left|w_B - w_B^*\right| + \left|w_C - w_C^*\right| \le Turnover^*$$

$$w_S, w_B, w_C \ge 0$$

WHERE:

$\hat{r}_P$ = The Expected Net Return on the Portfolio

$\hat{R}_S, \hat{R}_B, \hat{R}_C$ = The Expected Returns for the Stock, Bond, and Cash Asset Classes

w_S, w_B, w_C = The portfolio weights for the Stock, Bond, and Cash Asset Classes

τ_S, τ_B, τ_C = The Transaction Cost for Changing a Stock, Bond, or Cash Position

w_S^*, w_B^*, w_C^* = The Starting Portfolio Weights for the Stock, Bond, and Cash Asset Classes

$\hat{\sigma}_P^2 = w_S^2 Var(\hat{R}_S) + w_B^2 Var(\hat{R}_B) + w_C^2 Var(\hat{R}_C) + 2 w_S w_B Cov(\hat{R}_S, \hat{R}_B) + 2 w_S w_C Cov(\hat{R}_S, \hat{R}_C) + 2 w_B w_C Cov(\hat{R}_B, \hat{R}_C)$
= The Expected Variance of the Portfolio

$\overline{\sigma}_P^2$ = The Maximum Portfolio Variance to be Tolerated

LEVR = The leverage factor of the portfolio (100% = unleveraged, more = leverage)

$Turnover^*$ = The Maximum Portfolio Turnover to be Tolerated

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CASE 4: BENCHMARK RELATIVE OPTIMIZATION

$$\underset{w_S, w_B, w_C}{\text{MAX}} \quad \hat{\alpha}_P = (w_S - \overline{w}_S)\hat{R}_S + (w_B - \overline{w}_B)\hat{R}_B + (w_C - \overline{w}_C)\hat{R}_C$$

Subject to:

$$\hat{\sigma}^2_{TE} \le \overline{\sigma}^2_{TE}$$

$$w_S + w_B + w_C \le LEVR$$

$$|w_S - \overline{w}_S| \le S^*_{tilt}, \quad |w_B - \overline{w}_B| \le B^*_{tilt}, \quad |w_C - \overline{w}_C| \le C^*_{tilt}$$

$$w_S, w_B, w_C \ge 0$$

WHERE :

$\hat{\alpha}_p = \hat{R}_p - \hat{\overline{R}}_P$ = Expected Portfolio Value Added over the Benchmark

$\overline{w}_S, \overline{w}_B, \overline{w}_C$ = The Benchmark Weights for the Stock, Bond, and Cash Asset Classes

$S^*_{tilt}, B^*_{tilt}, C^*_{tilt}$ = The Maximum Tilts to be Tolerated in the Stock, Bond, and Cash Asset Classes

$$\hat{\sigma}^2_{TE} = (w_S - \overline{w}_S)^2 Var(\hat{R}_S) + (w_B - \overline{w}_B)^2 Var(\hat{R}_B) + (w_C - \overline{w}_C)^2 Var(\hat{R}_C)$$
$$+ 2(w_S - \overline{w}_S)(w_B - \overline{w}_B) Cov(\hat{R}_S, \hat{R}_B) + 2(w_S - \overline{w}_S)(w_C - \overline{w}_C) Cov(\hat{R}_S, \hat{R}_C)$$
$$+ 2(w_B - \overline{w}_B)(w_C - \overline{w}_C) Cov(\hat{R}_B, \hat{R}_C)$$

= The Expected Tracking Error of the Portfolio to the Benchmark

$\overline{\sigma}^2_{TE}$ = The Maximum Tracking Error of the Portfolio to the Benchmark to be Tolerated

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CASE 5: BENCHMARK RELATIVE & TURNOVER SENSITIVE OPTIMIZATION

$$\underset{w_S, w_B, w_C}{\text{MAX}} \quad \hat{\alpha}_P = (w_S - \overline{w}_S)\hat{R}_S + (w_B - \overline{w}_B)\hat{R}_B + (w_C - \overline{w}_C)\hat{R}_C - |w_S - w_S^*| \cdot \tau_S - |w_B - w_B^*| \cdot \tau_B - |w_C - w_C^*| \cdot \tau_C$$

Subject to:

$$\hat{\sigma}_{TE}^2 \leq \overline{\sigma}_{TE}^2$$

$$(w_S - \overline{w}_S) + (w_B - \overline{w}_B) + (w_C - \overline{w}_C) \leq 0$$

$$|w_S - w_S^*| + |w_B - w_B^*| + |w_C - w_C^*| \leq Turnover^*$$

$$|w_S - \overline{w}_S| \leq S_{tilt}^*, \quad |w_B - \overline{w}_B| \leq B_{tilt}^*, \quad |w_C - \overline{w}_C| \leq C_{tilt}^*$$

$$w_S, w_B, w_C \geq 0$$

WHERE:

$\hat{\alpha}_p = \hat{R}_p - \hat{\overline{R}}_P$ = Expected Portfolio Value Added over the Benchmark

$\overline{w}_S, \overline{w}_B, \overline{w}_C$ = The Benchmark Weights for the Stock, Bond, and Cash Asset Classes

S_{tilt}^*, B_{tilt}^*, C_{tilt}^* = The Maximum Tilts to be Tolerated in the Stock, Bond, and Cash Asset Classes

$$\hat{\sigma}_{TE}^2 = (w_S - \overline{w}_S)^2 Var(\hat{R}_S) + (w_B - \overline{w}_B)^2 Var(\hat{R}_B) + (w_C - \overline{w}_C)^2 Var(\hat{R}_C) + 2(w_S - \overline{w}_S)(w_B - \overline{w}_B) Cov(\hat{R}_S, \hat{R}_B) + 2(w_S - \overline{w}_S)(w_C - \overline{w}_C) Cov(\hat{R}_S, \hat{R}_C) + 2(w_B - \overline{w}_B)(w_C - \overline{w}_C) Cov(\hat{R}_B, \hat{R}_C)$$

= The Expected Tracking Error of the Portfolio to the Benchmark

$\overline{\sigma}_{TE}^2$ = The Maximum Tracking Error of the Portfolio to the Benchmark to be Tolerated

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CASE 1: BASIC OPTIMIZATION

NUMERICAL EXAMPLE

PROBLEM: Find the optimal allocation to stocks, bonds, and cash given the following expected returns and monthly return covariance matrix. The expected annualized volatility of the portfolio should not exceed 10%. No leverage is permitted.

Asset Forecasts	
Asset Class	Expected Returns
Stocks	1.6995
Bonds	0.3939
Cash	0.4026

RETURN COVARIANCE MATRIX (calculated based on monthly returns from Jan 95 to Dec 99)			
	Stocks	**Bonds**	**Cash**
Stocks	15.598	1.121	-0.004
Bonds	1.121	1.451	0.011
Cash	-0.004	0.011	0.001

ANSWER: A 10% annual volatility is a 100% annual variance. A 100% annual variance is a 100%/12 = 8.33333% monthly variance. The objective function is:

$$\text{MAX } R_P = 1.6995W_S + 0.3939W_B + 0.4026W_C$$

subject to a monthly variance not to exceed 8.3333% and W_S, W_B, and W_C all non-negative. The optimal solution is:

$W_S = 72.3\%$	$W_B = 0.0\%$	$W_C = 27.7\%$

With a monthly return of 1.3399% and a monthly variance of 8.3333%.

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CASE 2: BASIC OPTIMIZATION With Hard Constraints

NUMERICAL EXAMPLE

PROBLEM: Find the optimal allocation to stocks, bonds, and cash given the following expected returns and monthly return covariance matrix. The expected annualized volatility of the portfolio should not exceed 10%. Stocks cannot exceed 65% of the portfolio. The portfolio needs to generate an annual income yield of 3%. No leverage.

Asset Forecasts		
Asset Class	Expected Returns	Income Yield
Stocks	1.6995	0.1652
Bonds	0.3939	0.4868
Cash	0.4026	0.0000

RETURN COVARIANCE MATRIX (calculated based on monthly returns from Jan 95 to Dec 99)			
	Stocks	**Bonds**	**Cash**
Stocks	15.598	1.121	-0.004
Bonds	1.121	1.451	0.011
Cash	-0.004	0.011	0.001

ANSWER: A 3% annualized income yield is a 0.2466% monthly income yield. The optimization formulation is:

$$\text{MAX } R_P = 1.6995W_S + 0.3939W_B + 0.4026W_C$$

subject to

$$\text{Variance} < 8.3333\%$$
$$W_S < 65\%$$
$$0.1652W_S + 0.4868W_B + 0.0000W_C > 0.2466\%$$
$$W_S, W_B, W_C > 0$$

The optimal solution is:

$W_S = 65.0\%$	**$W_B = 28.6\%$**	**$W_C = 6.4\%$**

With a monthly return of 1.243%, a monthly variance of 8.3333% and a monthly income yield of 0.2466%.

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CASE 3: TURNOVER SENSITIVE OPTIMIZATION

NUMERICAL EXAMPLE

PROBLEM: Find the optimal allocation to stocks, bonds, and cash given the following expected returns and monthly return covariance matrix. The expected annualized volatility of the portfolio should not exceed 10%. Your current allocation is 60% stocks, 25% bonds, 15% cash. Annualized turnover less than 200%. No leverage is permitted.

Asset Return and Cost Forecasts		
Asset Class	Expected Returns	Turnover Costs
Stocks	1.6995	0.2000
Bonds	0.3939	0.1000
Cash	0.4026	0.5000

RETURN COVARIANCE MATRIX (calculated based on monthly returns from Jan 95 to Dec 99)			
	Stocks	**Bonds**	**Cash**
Stocks	15.598	1.121	-0.004
Bonds	1.121	1.451	0.011
Cash	-0.004	0.011	0.001

ANSWER: The optimization formulation is:

$$\text{MAX } R_P = 1.6995W_S + 0.3939W_B + 0.4026W_C - 0.20x|W_S - 0.65| - 0.10x|W_B - 0.25| - 0.05x|W_C - 0.15|$$

subject to:

$$\text{Variance} < 8.3333\%$$

$$|W_S - 0.65| - 0.10x|W_B - 0.25| - 0.05x|W_C - 0.15| < (200\%/12)$$

$$W_S + W_B + W_C < 100\%$$

$$W_S, W_B, W_C > 0$$

The optimal solution is:

W_S = 68.3%	**W_B = 25.0%**	**W_C = 6.7%**

With a monthly gr6ss return of 1.287%, a monthly net return of 1.266%, an annualized volatility of 9.75%, and a monthly turnover of 17% (200% annualized).

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CASE 4: BENCHMARK RELATIVE OPTIMIZATION

NUMERICAL EXAMPLE

PROBLEM: Find the optimal allocation to stocks, bonds, and cash given the following expected returns and monthly return covariance matrix. Your benchmark is 60% stocks, 40% bonds. The expected annualized tracking error should not exceed 2%. No leverage should be used. Stocks can deviate up to 10% from benchmark, while bonds and cash can deviate up to 25% from benchmark.

Asset Forecasts	
Asset Class	Expected Returns
Stocks	1.6995
Bonds	0.3939
Cash	0.4026

RETURN COVARIANCE MATRIX (calculated based on monthly returns from Jan 95 to Dec 99)			
	Stocks	**Bonds**	**Cash**
Stocks	15.598	1.121	-0.004
Bonds	1.121	1.451	0.011
Cash	-0.004	0.011	0.001

ANSWER: A 2% annual tracking error is a 4% annual tracking variance. A 4% annual tracking variance is a 4%/12 = 0.3333% monthly tracking variance. The objective fn is:

$$\text{MAX } \alpha_P = 1.6995(W_S - 0.60) + 0.3939(W_B - 0.40) + 0.4026(W_C - 0.00)$$

Subject to:

$$\text{Tracking Variance} < 0.3333\%$$
$$W_S + W_B + W_C < 100\%$$
$$|W_S - 0.60| < 10\%, \quad |W_B - 0.40| < 25\%, \quad |W_C - 0.00| < 25\%,$$
$$W_S, W_B, W_C > 0$$

The optimal solution is:

W_S = 70.0%	**W_B = 15.0%**	**W_C = 15.0%**

With a monthly portfolio expected return of 1.309% (0.132% over the expected benchmark return of 1.177%), an annualized tracking error of 1.52%, and a total expected portfolio volatility of 2.81% per month (9.74% per annum).