ANALYTICinvestors

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A GUIDE TO ANALYTIC'S BOND OPTIMIZATION PROGRAM

Overview

The Bond Optimization Program provides a framework for several different approaches to bond portfolio management, including finding the weights for a fixed income portfolio that:

(1) Minimizes tracking error of the portfolio relative to a stipulated bond index
(2) Maximizes portfolio return subject to various user-defined risk constraints
(3) Maximizes portfolio return subject to replication of a bond index's key characteristics

The Bond Optimization Program can constrain a portfolio as to its modified duration, convexity, average credit exposure, category of credit exposure, average maturity, average coupon, callability exposure, exposure to premium bonds, exposure to non-US instruments, and sector exposures. The objective can be to seek out those securities with the highest yield, highest return, or highest spread over a comparable Treasury that also fit the constraints.

An added feature of the Bond Optimization Program is its use of fixed income futures. By allowing the portfolio to employ any of a number of different short fixed income futures contracts to reduce duration, higher yielding bonds can be used in the portfolio than would be the case if the portfolio was constrained to hold only long spot market fixed income instruments. To the extent that wider yield spreads are associated with longer duration instruments (both because of liquidity preference and the positive slope of the credit spread curve), the use of fixed income futures allows the portfolio to hold such higher yielding instruments without taking duration risk in excess of that found in the benchmark index.

Setting the Scale of the Problem

Based on the variable definitions list in the Appendix, before we discuss the objective function or constraint set it is useful to set the scale of the optimization. There are two constraints that help define the scale of the problem, a Budget Constraint and a Notional Constraint:

$$\sum_{i\in B}(10000\times w_i P_i)+\sum_{j\in F} w_j F_j+(1000000\times w_c)=C^* \quad \text{(Budget, or Capital Constraint)}$$

$$\sum_{i\in B}(10000\times w_i)+(1000000\times w_c)=N^* \quad \text{(Notional Constraint)}$$

The *Budget* Constraint allocates the initial capital given by the client (or, if a portfolio is already in place, the market liquidation value of the existing portfolio) across three items: the bonds that could be purchased (weighted by their purchase price), a pool of cash valued at face value, and the initial margin for the short futures contracts. The constraint is scaled to be in dollars. The *Notional* Constraint defines the total notional (i.e. par) value of the portfolio as the sum of the par value of all the bonds bought plus the face value of cash held. The Notional constraint is also scaled to be in dollars. Note that the Notional Constraint is not affected by the holding of fixed income futures, since futures are purchased or sold on margin and do not return a par value at maturity. Futures also do not affect the portfolio's average coupon or average maturity for similar reasons. Futures do, however, provide total return in the form of cash flow streams, and as such they can affect such measures as yield, duration, and convexity.

As an example of a difference between the Budget Constraint and the Notional Constraint, let C* (the amount of capital given by the client with the intention of investing in a fixed income portfolio) be $10

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million. Assume the optimizer chooses to invest in a set of fixed income securities with an average invoice price of $80 per $100 of par value. Assume also for the sake of simplicity that the optimizer did not choose to short any futures contracts. The initial *market* value of the portfolio would then be $10 million, whereas the initial *par* value of the portfolio would be $100/$80 = $12.5 million. It is assumed that the client would be more concerned with C* (the amount of capital put up for investment) than with N* (the amount of notional par value bonds bought), so long as the constraint set kept the resulting portfolio within acceptable risk parameters. Unless a client wants to constrain the portfolio both as to capital invested and par value invested, only the Budget Constraint actually needs to be put directly in the optimizer as a separate constraint. The Notional Constraint will not appear in the optimizer as a separate constraint, but as described below it will be used algebraically to re-arrange some of the constraints.

Creation of the Objective Function

The objective function of the Bond Optimization Program seeks to maximize the weighted average yield of the final optimal portfolio. Given the manner in which the weights are defined in the Appendix, three issues arise in the creation of the objective function:

1) Cash – There is a decision variable for the holding of cash (w_c) in the objective function. This variable would be presumed to earn the STIF rate (or possibly to earn zero, or whatever rate overnight money can earn for the client), since there are otherwise specific decision variables included in the optimizer to allow for the purchase of Treasury bills and other discount instruments. Cash is distinguished from other short-term investments the optimizer might consider as that asset with a near-zero maturity.

2) Weights on the Bonds – The weights on the bonds (w_i) represent the number of $1 million par value bonds of type i purchased. Each weight is multiplied by the yield of the associated bond in the objective function. The yields on the bonds used in the objective function (y_i) represent the "true" yields, i.e., they are holiday- and weekend-adjusted yield-to-worst calculations.

3) Weights on the Futures – The weights on the futures contracts (w_j) represent the *number* of futures contracts of type j that are shorted in the portfolio. Constraints will be placed on the optimization to insure that only short holdings of futures are ever permitted, so that futures are used strictly to reduce risk. Note that where necessary the weights (w_j) will be adjusted by any necessary constant (M_j) to insure that the futures contracts positions represent a notional amount ($1 million) similar to what the weight on each bond (w_i) would buy ($1 million) in notional terms. For example, if a long bond futures contract covers a notional $100,000 of an underlying bond, then $M_j = 10$ would be used in conjunction with w_j to insure that the optimizer chooses a sufficient number achieve comparability. The weights on the futures are multiplied by the return *premium* (y_j-y_C) for the futures contract. The return premium on the futures contract is the yield on the futures contract minus the cash rate. This is the appropriate measure since the fixed income futures contracts are not being equitized with any specific pool of cash, and as such will not earn the full yield of the underlying instrument but only the return premium.

The full technical representation of the objective function would thus be:

$$MAX\ PORTYLD = \sum_{i \in B} w_i\, y_i - \sum_{j \in F} w_j \frac{(y_j - y_c)}{M_j} + w_c\, y_c$$

The user of the Bond Optimization Program must simply remember, however, that the value for PORTYLD that comes out of the optimizer must be divided by the total par value (N^*) of the portfolio in order for PORTYLD to represent the weighted average yield of the portfolio.

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Range Constraints

The range constraints seek to keep various aggregate risk and return portfolio statistics within certain ranges. The four main statistics to be constrained are:

(1) Par-value-weighted averaged modified duration.
(2) Par-value-weighted average convexity.
(3) Par-value-weighted average annual coupon rate.
(4) Par-value-weighted average time to maturity.

Maintaining the portfolio within such bands can be useful when trying to have the portfolio replicate an index (e.g., LB Govt/Corp), but achieve a higher average yield.

The range constraints are used in pairs, one to constrain the portfolio to *exceed* the minimum acceptable average of a given statistic and the other to constrain the portfolio to stay *beneath* a maximum acceptable average of a given statistic. If any constraint is not desired (e.g., shorter duration might always be acceptable and no minimum duration would be necessary), then the bound on that constraint can be set appropriately. As another example, if the portfolio wanted to have as large an average coupon as possible, the minimum coupon floor could be set fairly high, while the maximum coupon ceiling could be set to a sufficiently large value (i.e., larger than the largest coupon in the security universe) to insure that the average coupon was unbounded from above.

The use of decision variables that are expressed in numbers of bonds and numbers of futures contracts (rather than percent-of-notional weights in each) complicates slightly the algebra of the constraint formula. The weighted exposure to each characteristic must be divided by the notional amount of the portfolio. As an example, the following would be the constraint to insure that the portfolio's weighted average modified duration did not exceed a maximum level, given by D_U^*:

$$\frac{\sum_{i \in B} w_i D_i - \sum_{j \in F} w_j M_j D_j + w_c D_c}{N^*} \leq D_U^* \quad \text{(Upper Average Duration)}$$

By assuming that the duration of cash is zero, by substituting for N^* the full definition of the Budget Constraint, and after a bit of algebraic re-arranging this constraint can be made into a consistently linear constraint for purposes of linear programming by rewriting it as:

$$\sum_{i \in B} w_i \left(D_i - D_U^*\right) - \sum_{j \in F} w_j M_j D_j - w_c D_U^* \leq 0 \quad \text{(Upper Average Duration)}$$

The pair of constraints that would set a binding range around the portfolio's weighted average modified duration would then be written:

$$\sum_{i \in B} w_i \left(D_i - D_U^*\right) - \sum_{j \in F} w_j M_j D_j - w_c D_U^* \leq 0 \quad \text{(Upper Average Duration)}$$
$$\sum_{i \in B} w_i \left(D_i - D_L^*\right) - \sum_{j \in F} w_j M_j D_j - w_c D_L^* \geq 0 \quad \text{(Lower Average Duration)}$$

When taken as a group, the four pairs of constraints needed to provide the bounds on average duration, convexity, coupon, and maturity would be written as:

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$$\sum_{i\in B} w_i\left(D_i - D_U^*\right) - \sum_{j\in F} w_j M_j D_j - w_c D_U^* \le 0 \quad \text{(Upper Average Duration)}$$

$$\sum_{i\in B} w_i\left(D_i - D_L^*\right) - \sum_{j\in F} w_j M_j D_j - w_c D_L^* \ge 0 \quad \text{(Lower Average Duration)}$$

$$\sum_{i\in B} w_i\left(C_i - C_U^*\right) - \sum_{j\in F} v_j M_j C_j - w_c C_U^* \le 0 \quad \text{(Upper Average Convexity)}$$

$$\sum_{i\in B} w_i\left(C_i - C_L^*\right) - \sum_{j\in F} v_j M_j C_j - w_c C_L^* \ge 0 \quad \text{(Lower Average Convexity)}$$

$$\sum_{i\in B} w_i\left(CP_i - CP_U^*\right) - w_c CP_U^* \le 0 \quad \text{(Upper Average Coupon)}$$

$$\sum_{i\in B} w_i\left(CP_i - CP_L^*\right) - w_c CP_L^* \ge 0 \quad \text{(Lower Average Coupon)}$$

$$\sum_{i\in B} w_i\left(M_i - M_U^*\right) - w_c M_U^* \le 0 \quad \text{(Upper Average Maturity)}$$

$$\sum_{i\in B} w_i\left(M_i - M_L^*\right) - w_c M_L^* \ge 0 \quad \text{(Lower Average Maturity)}$$

Note that the last two pairs (for coupon and maturity) do not include terms for the futures contracts, since the futures contracts are not assumed to impact the average coupon or average maturity of the portfolio.

Rating and Sector Constraints

Two additional characteristics that need to be bounded in the final portfolio are the average credit rating and the various sector exposures. Average credit rating is based on each instrument's S&P rating. Each credit rating is assigned a numerical strength, and the par-value weighted average of these numerical strengths serves as the measure to be bounded. The numerical strengths associated with each S&P rating are:

Credit Category	Value
US Treasuries	0.0
AAA-rated	1.0
AA-rated	2.0
A-rated	3.0
BBB-Rated	4.0
BB-Rated	5.0
B-Rated	6.0

If a client account does not allow speculative grade bonds (below BBB) the average credit constraint does not specifically exclude those categories; however, a credit specific constraints discussed below can prevent their inclusion. Note also that the futures contracts are assumed to have a credit rating equivalent to that of U.S. Treasuries (0.0) and will not improve or reduce the average credit rating in the constraints if included, so they are simply excluded. Cash is excluded for similar reasons. Given these restrictions, the upper bound on the average credit rating of the portfolio can be written as:

$$\frac{\sum_{i\in B} w_i ARAT_i}{N^*} \le ARAT_U^* \quad \text{(UpperAverageCredit)}$$

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Re-arranging terms and allowing for a lower as well as an upper bound, we get the following dual constraints to bind the average credit rating of the portfolio:

$$\sum_{i \in B} w_i \left(ARAT_i - ARAT_U^* \right) - w_c ARAT_U^* \leq 0 \quad \text{(Upper Average Credit Rating)}$$
$$\sum_{i \in B} w_i \left(ARAT_i - ARAT_L^* \right) - w_c ARAT_L^* \geq 0 \quad \text{(Lower Average Credit Rating)}$$

If preferred, the portfolio can be restricted as to the specific percentages held in each of the different credit categories. For example, to provide upper and lower bounds on the holdings of AA and BBB bonds the constraints would be written as:

$$\sum_{i \in B} w_i \left(AA_i - AA_U^* \right) - w_c AA_U^* \leq 0 \quad \text{(Upper AA Holdings)}$$
$$\sum_{i \in B} w_i \left(AA_i - AA_L^* \right) - w_c AA_L^* \geq 0 \quad \text{(Lower AA Holdings)}$$
$$\sum_{i \in B} w_i \left(BBB_i - BBB_U^* \right) - w_c BBB_U^* \leq 0 \quad \text{(Upper BBB Holdings)}$$
$$\sum_{i \in B} w_i \left(BBB_i - BBB_L^* \right) - w_c BBB_L^* \geq 0 \quad \text{(Lower BBB Holdings)}$$

A separate constraint of each type can be developed for all possible rating categories (AAA, AA, A, BBB, BB, B, etc.). Each of these constraints can also be used to exclude any holding of an unacceptably low credit category. For example, we could exclude all holdings of BB rated bonds by setting BB_U^* equal to zero. Categories can also be grouped to provide constraints on holdings of investment grade, high grade, etc, by defining grouped variables such as HIGR (for high grade) and INVGR (for investment grade) and setting them equal to the sum of all the bonds that have ratings in those specific categories.

As for the sector constraints, these can vary depending on what sectors the user wants to define, such as corporate, agency, Treasury, Eurobond, convertible, etc. If we let $TREAS_i$ be 1 if bond i is a US Treasury and 0 if it is not, then the binding constraints for Treasuries can be written as:

$$\sum_{i \in B} w_i \left(TREAS_i - TREAS_U^* \right) - w_c TREAS_U^* \leq 0 \quad \text{(Upper Treasury Holdings)}$$
$$\sum_{i \in B} w_i \left(TREAS_i - TREAS_L^* \right) - w_c TREAS_L^* \geq 0 \quad \text{(Lower Treasury Holdings)}$$

This process can be repeated for as many different sectors as the user wants to define.

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Appendix - Definitions of Optimizer Variables

N^* = The total cash available for investment in bonds and/or futures, in millions
C^* = The total notional value of the portfolio, in millions
w_i = # of $1 million par bonds of type i to be bought
w_c = The total face value of assets held as cash, in millions
v_j = # of fixed income futures contracts of type j to be sold
M_j = Multiplier giving the number of fixed income futures per $1 million par value
B = An index set over which i ranges that contains all bonds in the security universe
F = An index set over which j ranges that contains all relevant fixed income futures
RAT_i = The numerical value given for the i^{th} bond's S&P rating (AAA=1, AA=2, etc.)
AAA_i = 1 if bond i has an AAA rating, 0 otherwise (repeated for AA, A, BBB, BB, B)
$HIGR_i$ = 1 if bond i is a high grade bond (AAA or AA), 0 otherwise
$INVGR_i$= 1 if bond i is an investment grade bond (BBB or better), 0 otherwise
$ZERO_i$ = 1 if bond i is a zero coupon bond, zero otherwise
$CALL_i$ = 1 if bond i is a callable bond, zero otherwise
$SECT_i$ = 1 if bond i is in the given sector (Treasury, agency, corporate, etc), 0 otherwise
D_i, D_j = The modified duration of bond i or future j, as appropriate
CP_i = The annual coupon paid on bond i (per $100 of par value)
C_i, C_j = The convexity of bond i and future j, as appropriate
P_i = The full invoice price of bond I (per $100 of par value)
F_j = The cash necessary for initial margin on the bond futures contract j (in dollars)
y_i,y_j,y_c = The yield of bond i, future j, or cash, respectively
T_i = The number of years to final maturity for bond i
D_u^*, D_l^*= The upper and lower bounds for the weighted average modified duration
CP_u^*, CP_l^* = The upper and lower bounds for the weighted average coupon
C_u^*,C_l^*=The upper and lower bounds for the weighted average convexity
M_u^*, M_l^* = The upper and lower bounds for the weighted average years to maturity

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The Full Formulation

$$MAX\ \ PORTYLD = \sum_{i\in B} w_i y_i - \sum_{j\in F} v_j M_j y_j + w_c y_c$$

$$S.T.$$

$$\sum_{i\in B} w_i P_i + \sum_{j\in F} v_j M_j F_j + w_c = N^* \quad \text{(Budget Constraint)}$$

$$\sum_{i\in B} w_i (D_i - D_U^*) - \sum_{j\in F} w_j M_j D_j - w_c D_U^* \le 0 \quad \text{(Upper Average Duration)}$$

$$\sum_{i\in B} w_i (D_i - D_L^*) - \sum_{j\in F} w_j M_j D_j - w_c D_L^* \ge 0 \quad \text{(Lower Average Duration)}$$

$$\sum_{i\in B} w_i (C_i - C_U^*) - \sum_{j\in F} w_j M_j C_j - w_c C_U^* \le 0 \quad \text{(Upper Average Convexity)}$$

$$\sum_{i\in B} w_i (C_i - C_L^*) - \sum_{j\in F} w_j M_j C_j - w_c C_L^* \ge 0 \quad \text{(Lower Average Convexity)}$$

$$\sum_{i\in B} w_i (CP_i - CP_U^*) - w_c CP_U^* \le 0 \quad \text{(Upper Average Coupon)}$$

$$\sum_{i\in B} w_i (CP_i - CP_L^*) - w_c CP_L^* \ge 0 \quad \text{(Lower Average Coupon)}$$

$$\sum_{i\in B} w_i (M_i - M_U^*) - w_c M_U^* \le 0 \quad \text{(Upper Average Maturity)}$$

$$\sum_{i\in B} w_i (M_i - M_L^*) - w_c M_L^* \ge 0 \quad \text{(Lower Average Maturity)}$$

$$\sum_{i\in B} w_i (ARAT_i - ARAT_U^*) - w_c ARAT_U^* \le 0 \quad \text{(Upper Average Rating)}$$

$$\sum_{i\in B} w_i (ARAT_i - ARAT_L^*) - w_c ARAT_L^* \ge 0 \quad \text{(Lower Average Rating)}$$

$$\sum_{i\in B} w_i (RAT_i - RAT_U^*) - w_c RAT_U^* \le 0 \quad \text{(Upper Percentage Holding of Rating Type)}$$

$$\sum_{i\in B} w_i (RAT_i - RAT_L^*) - w_c RAT_L^* \ge 0 \quad \text{(Lower Percentage Holding of Rating Type)}$$

$$\sum_{i\in B} w_i (SECT_i - SECT_U^*) - w_c SECT_U^* \le 0 \quad \text{(Upper Percentage Holding of Rating Type)}$$

$$\sum_{i\in B} w_i (SECT_i - SECT_L^*) - w_c SECT_L^* \ge 0 \quad \text{(Lower Percentage Holding of Rating Type)}$$

$$\sum_{i\in B} w_i (CALL_i - CALL_U^*) - w_c CALL_U^* \le 0 \quad \text{(Upper Percentage Holding of Callables)}$$

$$\sum_{i\in B} w_i (CALL_i - CALL_L^*) - w_c CALL_L^* \ge 0 \quad \text{(Lower Percentage Holding of Callables)}$$

$$\sum_{i\in B} w_i (ZERO_i - ZERO_U^*) - w_c ZERO_U^* \le 0 \quad \text{(Upper Percentage Holding of Zeros)}$$

$$\sum_{i\in B} w_i (ZERO_i - ZERO_L^*) - w_c ZERO_L^* \ge 0 \quad \text{(Lower Percentage Holding of Zeros)}$$

SECT = AGCY, CONV, CORPF, CORPI, EMERG, EURO, MTN, STRIP, SUPR, TREAS, YANKEE
RAT = AAA, AA, A, BBB, BB, B